INTRINSIC EQUATIONS OF ROTATIONAL GAS FLOWS

BY

E. R. SURYANARAYAN

ABSTRACT

Considering a vortex line as a C^3 curve in E^3, equations governing the flow of a steady, compressible gas are expressed in the intrinsic form. These intrinsic relations are applied to derive some geometric properties of rotational motions, and to study a class of flows whose vortex lines form a family of helices on right circular cylinders.

1. **Introduction.** Coburn [1], Kanwal [2], Wasserman [3] and Truesdell [4] have derived the intrinsic form of the equations governing the steady motion of a gas, by considering a stream line as a space curve in in E^3, a three-dimensional Eucldean space. In this paper we derive the equations of motion, continuity and entropy of a rotational flow in the intrinsic form by considering a vortex line as a C^3 curve in E^3. It is assumed the gas is non-viscous and non-heat-conducting.

By expressing the equations of motion along the tangent, principal normal and the binormal vectors of the vortex line, it is shown that for a circulation preserving complex-lemellar flow the Lamb surfaces and the surfaces normal to the stream lines intersect orthogonally along the vortex lines, and therefore, if the vortex lines are lines of curvature on either the Lamb surface or the surface orthogonal to the stream lines, then they are lines of curvature on the other also. It is found that in a circulation preserving motion, a vortex line is a geodesic on the Lamb surface if and only if the velocity component along the principal normal vector is zero. A necessary and sufficient condition is determined for the Lamb surfaces to be a family of parallel surfaces. It is shown that if the vortex line admits normal surfaces then these surfaces are minimal if and only if the magnitude of the vorticity vector does not vary along the vortex line.

The intrinsic equations are used to derive the flow equations when the vortex-lines are right circular helices. A class of solutions of these equations are obtained in the case when the binormals of these helices form a normal congruence.

2. **The basic equations.** Let $x^j (j = 1,2,3)$ denote a Cartesian orthogonal coordinate system in E^3 and let us denote the partial derivative by the symbolism

$$\partial_j \equiv \frac{\partial}{\partial x^j}.$$

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In E^3 covariant and contravariant components are equivalent. However, in order to use the summation convention, we shall write the indices in covariant and contravariant positions. Let g_{ij} denote the fundamental tensor of E^3.

The equations governing the flow of a stationary gas, neglecting viscosity and thermal conductivity, are

$$\partial_j(\rho u^j) = 0, \tag{2.1}$$

$$\rho u^j \partial_j u_i + \partial_i p = 0, \tag{2.2}$$

$$u^j \partial_j \eta = 0, \tag{2.3}$$

$$\rho = P(p)\, S(\eta), \tag{2.4}$$

where, u_j are the components of the velocity, ρ is the density, p is the pressure, and η denotes the specific entropy. For a polytropic gas

$$P(p) = p^{1/\gamma}, \tag{2.5}$$

γ being the adiabatic exponent.

If e^{ijk} denotes the permutation tensor, then the vorticity vector is defined by

$$\omega^i = e^{ijk} \partial_j u_k. \tag{2.6}$$

The Lamb vector λ_i is defined by

$$\lambda_i = e_{ijk} \omega^j u^k. \tag{2.7}$$

The equations of motion (2.2) can be written in terms of the Lamb vector [5]

$$\lambda_i = T \partial_i \eta - \partial_i H \tag{2.8}$$

where, T is the absolute temperature and H is the stagnation enthalpy defined by

$$H = \frac{1}{2} u^2 + I. \tag{2.9}$$

Here u is the magnitude of u_i and I is the specific enthalpy. A consequence of the relation (2.6) is

$$\partial_j \omega^j = 0. \tag{2.10}$$

3. **The basic decomposition.** If s^j is a unit vector along the direction of ω^j, then we may write

$$\omega^j = \omega s^j \tag{3.1}$$

where ω is the magnitude of ω^j. Let n^j and b^j be the unit vectors along the principal normal vector and the binormal vector respectively, of the vortex line. The velocity vector can be expressed in terms of s^j, n^j and b^j:

(3.2) $$u^j = \alpha_1 s^j + \alpha_2 n^j + \alpha_3 b^j.$$

Substituting for ω^j from (3.1) and for u^j from (3.2) into (2.8) and using the cross-product relations of s^j, n^j ans b^j, (2.8) becomes

(3.3) $$\omega(\alpha_2 b_i - \alpha_3 n_i) = T\partial_i \eta - \partial_i H.$$

Decomposing the right hand side of the above relation along s_i, n_i and b_i and equating their coefficients respectively, we get

(3.4) $$\frac{dH}{ds} = 0,$$

(3.5) $$T\frac{d\eta}{dn} - \frac{dH}{dn} = -\alpha_3\omega,$$

(3.6) $$T\frac{d\eta}{db} - \frac{dH}{db} = \alpha_2\omega,$$

where, d/ds, d/dn and d/db denote the directional derivatives operator along the directions s^j, n^j and b^j respectively. The equation of continuity (2.1), by use of (2.3), (2.4) and (3.2), becomes

(3.7)
$$\left(\alpha_1\frac{d}{ds} + \alpha_2\frac{d}{dn} + \alpha_3\frac{d}{db}\right)P + P\left(\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \frac{d\alpha_1}{ds} + \frac{d\alpha_2}{dn} + \frac{d\alpha_3}{db}\right) = 0,$$

where

(3.8) $$A_1 = \partial_i s^i, \qquad A_2 = \partial_i n^i, \qquad A_3 = \partial_i b^i.$$

The equation (2.3) can be written

$$\left(\alpha_1\frac{d}{ds} + \alpha_2\frac{d}{dn} + \alpha_3\frac{d}{db}\right)\eta = 0.$$

Substituting into (2.6) for ω^i from (3.1) and for u^i from (3.2), and equating the coefficients of s^i, n^i and b^i we find after a lengthy but direct computation that

(3.9) $$\alpha_1\Omega_1 + \alpha_2 d - \alpha_3 t + \frac{d\alpha_3}{dn} - \frac{d\alpha_2}{db} = \omega,$$

(3.10) $$\alpha_2\Omega_2 + \alpha_3 q - \frac{d\alpha_3}{ds} + \frac{d\alpha_1}{db} = 0,$$

(3.11) $$\alpha_3\Omega_3 - \alpha_2 e + k\alpha_1 + \frac{d\alpha_2}{ds} - \frac{d\alpha_1}{dn} = 0,$$

where Ω_1, Ω_2, Ω_3 are rotation coefficients [6] defined by

(3.12) $$\Omega_1 = b^k\frac{ds_k}{dn} - n^k\frac{ds_k}{db}, \quad \Omega_2 = b^k\frac{dn_k}{ds} - s^k\frac{dn_k}{db}, \quad \Omega_3 = n^k\frac{db_k}{ds} - s^k\frac{db_k}{dn}$$

k is the curvature of the vortex lines, and

$$d = \frac{dn_k}{dn} b^k, \quad e = \frac{dn_k}{dn} s^k, \quad q = \frac{db_k}{db} s^k, \quad t = \frac{db_k}{db} n^k. \tag{3.13}$$

The relation (2.10), when substituted for ω^i from (3.1), yields

$$\frac{d}{ds}(\log \omega) = -A_1. \tag{3.14}$$

Let us now draw some conclusions from our calculations. The relation (3.4) shows that the surfaces $H =$ constant contain the vortex lines. This is a well known result [5]. Relations (3.5) and (3.6) show that the rate of change of H along the principal normal direction is completely determined by the rate of change of η along this direction, and similarly for the rate of change of H along the binormal direction.

For a circulation preserving motion, there exists an acceleration potential ϕ and therefore [4]

$$\lambda_i = -\partial_i(\phi + \frac{1}{2}u^2). \tag{3.15}$$

That is, ϕ satisfies the relation

$$\partial_i\phi = \partial_i I - T\partial_i\eta.$$

The surfaces $\phi + (1/2)u^2 =$ constant are the Lamb surfaces. From (3.15) it is clear that the Lamb surfaces contain both stream lines and vortex lines. If the motion is complex-lamellar, that is, if there exists a family of ∞^1 surfaces orthogonal to the stream lines, then these surfaces contain vortex lines. And these two families of surfaces, the Lamb surfaces and the surfaces orthogonal to the stream lines, intersect orthogonally. Therefore we have:

THEOREM. *In a circulation preserving complex-lamellar flow, the Lamb surfaces and the surfaces orthogonal to the stream lines intersect orthogonally along the vortex lines.*

This result is more general than the one in the paper [7]. Since the Lamb surfaces and the surfaces orthogonal to the stream lines are orthogonal we apply a classical theorem due to Joachimsthal [8] and obtain the following statement: *In a circulation preserving complex-lamellar motion if the vortex line is a line of curvature on either the Lamb surface or the surface orthogonal to the stream line, then it is a line of curvature on the other also.*

From the relation (2.8), (3.3) and (3.5) we find that the normal to the Lamb surface is along the principal normal of the vortex line if and only if $\alpha_2 = 0$. Therefore, *a vortex line is a geodesic on the Lamb surface in a circulation preserving motion, if and only if the component of the velocity vector along the principal normal vector of the vortex line is zero.*

From (2.8), (3.3) and (3.15) we find that

$$\left[\frac{d}{d\lambda}\left(\phi + \frac{1}{2}u^2\right)\right]^2 = \omega^2(\alpha_2^2 + \alpha_3^2)$$

where $d/d\lambda$ is the directional derivative along the Lamb vector. Therefore, a necessary and sufficient condition that the surfaces, containing u_i and ω_i, be parallel surfaces is that

$$\omega^2(\alpha_2^2 + \alpha_3^2) = a^2$$

where a is a constant along each surface of the family. A similar result has been obtained by Coburn [1]. The above condition is also equivalent to

$$\alpha_2 = \frac{a}{\omega}\cos\alpha, \quad \alpha_3 = \frac{a}{\omega}\sin\alpha,$$

α being a parameter.

The relation (3.14) shows that *the rate of change of* $-\log\omega$ *along the vortex lines equals the divergence of the vector* s^i. *If* $A_1 = \partial_i s^i = 0$, *then* ω *does not vary along the vortex lines, and conversely.* The condition $A_1 = 0$ is a geometrical condition on the congruence of vortex lines. It means, roughly, that the vortex lines do not converge (or diverge).

A motion for which the vortex lines possess a family of normal surfaces satisfies $\Omega_1 = 0$, and A_1 is then the mean curvature of these normal surfaces. In particular ω *does not vary along the vortex lines if and only if the mean curvature* A_1 *vanishes, that is, if and only if the family of normal surfaces constitute a family of minimal surfaces.* From the previous results in this section we find that, in a circulation preserving complex lamellar motion with vortex lines possessing normal surfaces, the velocity vector, the vorticity vector and the Lamb vector are mutually orthogonal at every point of the flow, that is, the Lamb surfaces, the surfaces orthogonal to the stream lines and the surfaces orthogonal to the the vortex lines are mutually orthogonal. This result is known [9].

For a screw motion, that is, for a motion with ω^i parallel to u^i, $\alpha_2 = \alpha_3 = 0$, $\alpha_1 = u$. Therefore (3.10) and (3.11) imply that u does not vary along the binormal of the vortex lines and that the variation of $\log u$ along the principal normal equals the curvature of the vortex lines.

4. **Flows whose vortex lines are right circular helices.** Here we shall consider the case where the vortex lines form a family of right circular helices. We introduce the cylindrical coordinates r, θ and z and write

$$s_i = \theta_i \sin\beta + z_i \cos\beta, \tag{4.1}$$

$$n^i = -r_i, \tag{4.2}$$

$$b_i = -\theta_i \cos\beta + z_i \sin\beta. \tag{4.3}$$

Here θ_i and z_i are the unit vectors along the increasing θ and z directions respectively and r_i is the unit vector along the radius of the cylinder. β is the angle of the helices and is in general a function of r. Since the vortex lines form geodesics on cylinders, we have $\alpha_2 = 0$. (Section 3).

Since helices form a congruence of parallel curves on cylinders, $A_1 = 0$ [10]. After some computation we find that:

$$A_1 = A_3 = d = e = q = 0, \tag{4.4}$$

$$k = \frac{\sin^2\beta}{r}, \quad t = \frac{\cos^2\beta}{r}, \tag{4.5}$$

$$\Omega_1 = \frac{d\beta}{dr} + \frac{\sin\beta\cos\beta}{r}, \ \Omega_2 = 0, \ \Omega_3 = -\frac{\sin\beta\cos\beta}{r} + \frac{d\beta}{dr} \tag{4.6}$$

and

$$\frac{d}{ds} = \frac{\sin\beta}{r}\frac{\partial}{\partial\theta} + \cos\beta\frac{\partial}{\partial z}, \ \frac{d}{dn} = -\frac{\partial}{\partial r}, \frac{d}{db} = \frac{\cos\beta}{r}\frac{\partial}{\partial\theta} - \sin\beta\frac{\partial}{\partial z}. \tag{4.7}$$

Substituting these relations into (3.4)–(3.7), (3.9)–(3.14) we find that

$$\frac{dH}{ds} = 0, \tag{4.8}$$

$$T\frac{\partial n}{\partial r} + \frac{\partial H}{\partial r} = -\alpha_3\omega \tag{4.9}$$

$$T\frac{d\eta}{db} - \frac{dH}{db} = 0 \tag{4.10}$$

$$\left(\alpha_1\frac{d}{ds} + \alpha_3\frac{d}{db}\right)P + P\left(\frac{d\alpha_1}{ds} + \frac{d\alpha_3}{db}\right) = 0, \tag{4.11}$$

$$-\alpha_3\frac{\cos^2\beta}{r} + \alpha_1\left(\frac{d\beta}{dr} + \frac{\cos\beta\sin\beta}{r}\right) - \frac{\partial\alpha_3}{\partial r} = \omega, \tag{4.12}$$

$$-\frac{d\alpha_3}{ds} + \frac{d\alpha_1}{db} = 0, \tag{4.13}$$

$$\alpha_3\left(-\frac{\sin\beta\cos\beta}{r} + \frac{d\beta}{dr}\right) + \frac{\sin^2\beta}{r}\alpha_1 + \frac{\partial\alpha_1}{\partial r} = 0, \tag{4.14}$$

$$\frac{d\omega}{ds} = 0. \tag{4.15}$$

The entropy condition (2.3) now becomes

$$\left(\alpha_1\frac{d}{ds} + \alpha_3\frac{d}{db}\right)\eta = 0. \tag{4.16}$$

In the above relations d/ds and d/db are given by (4.7). *The above equations constitute the flow equations for a motion whose vortex lines are right circular helices.*

5. **A special flow.** We shall now find a class of solutions of the set of equations (4.8)–(4.16). We shall assume that the flow is isentropic that is, $\eta \equiv$ constant, and the gas is polytropic. Further let

$$\frac{d\beta}{dr} - \frac{\cos\beta \sin\beta}{r} = 0 \tag{5.1}$$

$$\alpha_3 = \alpha_3(r),\ \alpha_1 = a_1(r),\ H = H(r),\ \omega = \omega(r),\ P = P(r). \tag{5.2}$$

The condition (5.1) implies that $\Omega_3 = 0$ and therefore b_i forms a normal congruence and

$$\tan\beta = r/\xi, \tag{5.3}$$

where ξ is an arbitrary constant. The equations (4.8)–(4.16), by virtue of (5.1)–(5.3), now reduce to

$$-\frac{dH}{dr} = \omega\alpha_3, \tag{5.4}$$

$$-\alpha_3 \frac{\cos^2\beta}{r} + \alpha_1\left(\frac{2\xi}{r^2 + \xi^2}\right) - \frac{d\alpha_3}{dr} = \omega, \tag{5.5}$$

$$\alpha_2 = 0, \tag{5.6}$$

$$\frac{d\alpha_1}{dr} = -\frac{r}{\xi^2 + r^2}\alpha_1. \tag{5.7}$$

Integrating equation (5.7), we get

$$\alpha_1 = A\xi(\xi^2 + r^2)^{-\frac{1}{2}}, \tag{5.8}$$

where A is an arbitrary constant. (5.3), (5.7) and (5.8) imply that

$$-\alpha_3 \frac{\xi^2}{r(\xi^2 + r^2)} + \frac{2A\xi^2}{(\xi^2 + r^2)} - \frac{d\alpha_3}{dr} = \omega. \tag{5.9}$$

In particular if

$$\alpha_3 = -Ar(\xi^2 + r^2)^{-\frac{1}{2}}, \tag{5.10}$$

the equation (5.9) reduces to

$$\omega = 4A\xi^2(\xi^2 + r^2)^{-\frac{3}{2}}. \tag{5.11}$$

Substituting for α_3 from (5.10) and for ω from (5.11) into (5.4) and integrating, we obtain

$$H = B^2 - \frac{2A^2\xi^2}{(\xi^2 + r^2)}. \tag{5.12}$$

Here B is an arbitrary constant. Since we have assumed the gas to be polytropic, the specific enthalpy [5]

$$I = \int_0^p \frac{dp}{\rho} = \int_0^p \frac{dp}{P} = \frac{\gamma}{S(\gamma - 1)} p^{(\gamma-1)/\gamma}, \tag{5.13}$$

where $S \equiv S(\eta) \equiv$ constant. From (5.8) and (5.10) the magnitude of the velocity is found to be constant and is equal to A. Therefore from (2.9), (5.12) and (5.13), we find that the pressure is given by

$$B^2 - \frac{2A^2\xi^2}{\xi^2 + r^2} - \tfrac{1}{2}A^2 = \frac{\gamma}{S(\gamma - 1)} p^{(\gamma-1)/\gamma}. \tag{5.14}$$

From (3.2), (5.8), (5.9), (4.1) and (4.3), the velocity vector becomes

$$u_i = \frac{A}{\xi^2 + r^2} [2\xi r \theta_i + (\xi^2 - r^2) z_i]. \tag{5.15}$$

For an isentropic motion of a polytropic gas, the equations of motion (2.2) and continuity (2.1) assume the form

$$Sp^{1/\gamma} u^j \partial_j u_i + \partial_i p = 0,$$
$$\partial_j (p^{1/\gamma} u^j) = 0.$$

Transforming the above equations into polar coordinates, we find that the components of velocity and pressure satisfy the transformed equations. From (5.15), we find that the stream lines are also right circular helices, making an angle δ with the z axis, where

$$\cos\delta = \frac{\xi^2 - r^2}{\xi^2 + r^2}.$$

As r changes from 0 to infinity δ changes from 0 to π. At $r = \xi$, $\delta = \pi/2$. The vorticity vector, from (3.1), (4.1) and (5.11), becomes

$$\omega^i = \frac{4A\xi^2}{(\xi^2 + r^2)^2} [r\theta_i + \xi z_i].$$

The angle β of the vortex lines changes from 0 to $\pi/2$ as r changes from 0 to infinity. We find that the angle between u^i and ω^i is also β.

Since [1]

$$I = \frac{c^2}{\gamma - 1}$$

where c is the local sound speed, the relation (5.13) and (5.14) yield

$$M^2 = \left(\frac{\xi^2 + r^2}{5\xi^2 + r^2}\right)\left(\frac{B^2}{c^2} - \frac{1}{\gamma - 1}\right).$$

Here $M = u/c$ is the Mach number. For real M the constant B must satisfy

$$B^2 > \frac{c^2}{\gamma - 1}.$$

The flow is supersonic, sonic or subsonic according as

$$\left(\frac{\xi^2 + r^2}{5\xi^2 + r^2}\right)\left(\frac{B^2}{c^2} - \frac{1}{\gamma - 1}\right) \gtreqless 1.$$

Other classes of flows may be found by assigning appropriate functions to α_3 in (5.9). Different classes of flows may be obtained by assigning appropriate values to β.

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University of Queensland (Australia)
University of Rhode Island (U. S. A.)